

Multiparticle Quantum Cosmology¹

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Abstract. Fock space quantization of Hamiltonian constraints of General Relativity and thermodynamics of quantum states for flat Friedmann–Lemaître–Robertson–Walker metrics is presented.

Keywords: quantum cosmology, quantum gravity, tachyon, Bose condensates, 2nd quantization, thermodynamics of quantum states, canonical Dirac–ADM formalism of General Relativity, MEQS

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INTRODUCTION. THERMODYNAMICAL EINSTEIN’S DREAM

General Relativity is a theory of some Riemannian manifold [1] with dynamics governs by the Einstein equations [2] that can be deduced from the Hilbert dynamical action [3] by the Palatini variational principle [4]

$$S_{\text{EH}} = \int d^4x \sqrt{-\det g_{\mu\nu}} \left(-\frac{1}{6}R + \mathcal{L} \right), \quad (1)$$

$$\frac{\delta S_{\text{EH}}}{\delta g^{\mu\nu}} = 0 \implies R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 3T_{\mu\nu}, \quad T_{\mu\nu} = \frac{2}{\sqrt{-\det g_{\mu\nu}}} \frac{\delta (\sqrt{-\det g_{\mu\nu}} \mathcal{L})}{\delta g^{\mu\nu}}, \quad (2)$$

where $R_{\mu\nu}$ is the Ricci tensor, $R = g^{\alpha\beta}R_{\alpha\beta}$ is the Ricci curvature scalar, \mathcal{L} is a full Lagrangian of all physical fields, and $T_{\mu\nu}$ is the stress–energy tensor of these fields³.

Let us to take up address an issue of a Riemannian manifold given by the flat Friedmann–Lemaître–Robertson–Walker metric [5] characterized by an interval⁴

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = (dx^0)^2 - a^2(t)(dx^i)^2, \quad (3)$$

where a is the Friedmann conformal scale factor and x^μ , $\mu = 0, 1, 2, 3$ is a cartesian coordinate system. In the canonical Dirac–ADM approach to General Relativity [6] the metric (3) is characterized by vanishing shift function and lapse function N is given by

$$d\eta = N(x^0)dx^0 \equiv \frac{dt}{a(t)}, \quad \text{where } t = \tau + x^0, \quad \tau = \text{const}, \quad (4)$$

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³ In this paper the units $8\pi G/3 = c = \hbar = k_B = 1$ are used.

⁴ For closed case $V_0 = \int d^3x < \infty$ the metrics (3) describe *cylindrical Einstein–Friedmann Universe*. [7]

where t is called cosmological time, and η is called conformal time. The metrics (3) in the Dirac-ADM approach are equivalent to the following Hamiltonian constraints

$$H = p_a^2 - 4V_0^2 a^4 H^2(a) = 0, \quad (5)$$

where p_a is canonical momentum conjugated to a , and $H(a)$ is the Hubble parameter,

$$p_a = -2V_0 \frac{da}{d\eta}, \quad H^2(a) \equiv \left(\frac{1}{a} \frac{da}{dt} \right)^2 = \left(\frac{1}{a^2} \frac{da}{d\eta} \right)^2 = \frac{1}{V_0} \int d^3x \mathcal{H}(x) \quad (6)$$

where $\mathcal{H}(x)$ is summarized Hamiltonian of physical fields associated by the Legendre transformation with the Lagrangian \mathcal{L} in the dynamical action (1). The Hubble law in the canonical approach is a result of integration of (5) and has a form

$$\int_{a_I}^a \frac{da'}{a' H(a')} = t_I - t, \quad \text{where } I \text{ means initial data.} \quad (7)$$

We propose call by the *Thermodynamical Einstein's Dream (TED)* a some formal road from General Relativity (1-2) to generalized thermodynamics of quantum states of a Riemannian manifold being a solution of this theory. The questions arise

- *Could TED really be realize?*
- *Can be exist a way that allows to get to know thermal properties of Spacetime?*
- *If these properties of Spacetime give new opportunities for Quantum Gravity?*

In this paper a proposal for realization of TED is demonstrated by using of second quantization in the Fock space and associated with this quantization formalism of the Bose condensation, where a *particle* is quantum state of a Spacetime, is discussed.

CONSTRAINTS, STRINGS, AND SECOND QUANTIZATION

Let us consider the Dirac-ADM primary constraints (5). Applying to this equation the formal identification

$$-4V_0^2 a^4 H^2(a) = m^2(a) \quad (8)$$

allows to equate the geometrical object (3) with free bosonic string with negative square of mass, called a *tachyon* [9], defined by the constraints

$$p_a^2 + m^2 = E^2, \quad E = 0, \quad m^2 = m^2(a) \leq 0. \quad (9)$$

The Hubble law (7) can be treated for tachyon as definition of spatial volume V_0

$$\left(\int_{a_I^2}^{a^2} \frac{dy}{|m(y)|} \right)^{-1} \Delta t = V_0, \quad \Delta t = t - t_I. \quad (10)$$

First quantization of the constraints (9) gives the Wheeler-DeWitt equation [10] for (3)

$$i[p_a, a] = 1 \implies \left[-\frac{\partial^2}{\partial a^2} + m^2(a) \right] \Psi = 0, \quad \Psi = \Psi^*, \quad (11)$$

that can be treated as the free-time Schrödinger equation [11]. However, this point of view has no physical sense. Equally well the Wheeler–DeWitt equation (11) can be understood as the 0+1 Klein–Gordon–Fock equation, that treats a Riemannian manifold as a Bose field. Separation of variables method applied to the equation (11) gives

$$(i\Gamma^\mu \partial_\nu - \mathbf{M}_\nu^\mu) \Phi_\mu = 0, \quad \mathbf{M}_\nu^\mu = \begin{bmatrix} 0 & -1 \\ -m^2 & 0 \end{bmatrix} \geq 0 \quad (12)$$

where matrices $\Gamma^\mu = [\mathbf{0}_2, i\mathbf{I}_2]$ create Clifford algebra $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}\mathbf{I}_2$, and was introduced the state vector $\Phi_\mu = [\Psi, \Pi_\Psi]^T$ with Π_Ψ being classical canonical momentum field conjugated to wave function Ψ , here also $\partial_\nu = [0, -\frac{\partial}{\partial a}]^T$. Let us carry second quantization [8] of the equation (12) that is agreed with (11). This has a proposed form

$$\begin{bmatrix} \hat{\Psi}[a] \\ \hat{\Pi}_\Psi[a] \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2|m(a)|} & 1/\sqrt{2|m(a)|} \\ -i\sqrt{|m(a)|}/2 & i\sqrt{|m(a)|}/2 \end{bmatrix} \begin{bmatrix} G[a] \\ G^\dagger[a] \end{bmatrix}, \quad (13)$$

and automatically fulfills principal rule of quantum field theory of the Bose systems

$$[\hat{\Pi}_\Psi[a], \hat{\Psi}[a']] = -i\delta(a - a'), \quad a \equiv a(\eta), \quad a' \equiv a(\eta') \quad (14)$$

and lies in accordance with general algebraic approach investigated in papers of Von Neumann, Araki and Woods [12]. In result, considered Bose system is described by the dynamical basis in the Fock space

$$B_a = \left\{ \begin{bmatrix} G[a] \\ G^\dagger[a] \end{bmatrix} : [G[a], G^\dagger[a']] = \delta(a - a'), [G[a], G[a']] = 0 \right\}, \quad (15)$$

according to the evolution equation

$$\frac{\partial B_a}{\partial a} = \begin{bmatrix} -im & \frac{1}{2m} \frac{\partial m}{\partial a} \\ \frac{1}{2m} \frac{\partial m}{\partial a} & im \end{bmatrix} B_a, \quad (16)$$

that can be diagonalized to the Heisenberg form by the Bogoliubov transformation

$$B'_a = \begin{bmatrix} u & v \\ v^* & u^* \end{bmatrix} B_a, \quad \frac{\partial B'_a}{\partial a} = \begin{bmatrix} -i\lambda & 0 \\ 0 & i\lambda \end{bmatrix} B'_a, \quad (17)$$

where $|u|^2 - |v|^2 = 1$, and

$$B'_a = \left\{ \begin{bmatrix} G'[a] \\ G'^\dagger[a] \end{bmatrix} : [G'[a], G'^\dagger[a']] = \delta(a - a'), [G'[a], G'[a']] = 0 \right\}, \quad (18)$$

is some new basis, and the Bogoliubov coefficients u and v are functions of a . This procedure gives simply $\lambda = 0$, and by this $G'[a] = w_I = \text{const}$ defines stable vacuum $|0\rangle_I = \left\{ |0\rangle_I : w_I |0\rangle_I = 0, 0 = {}_I\langle 0| w_I^\dagger \right\}$ in the static Fock basis $B'_a = B_I$,

$$B_I = \left\{ \begin{bmatrix} w_I \\ w_I^\dagger \end{bmatrix} : [w_I, w_I^\dagger] = 1, [w_I, w_I] = 0 \right\}. \quad (19)$$

As the result the system of equations for the Bogoliubov coefficients is obtained in the form

$$\frac{\partial}{\partial a} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -im & \frac{1}{2m} \frac{\partial m}{\partial a} \\ \frac{1}{2m} \frac{\partial m}{\partial a} & im \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}, \quad (20)$$

and by applying of the hyperbolic parametrization, this system can be solved directly as

$$u(a) = \frac{1}{2} \exp \left\{ \pm i \int_{a_I}^a m da \right\} \left(\sqrt{\left| \frac{m}{m_I} \right|} + \sqrt{\left| \frac{m_I}{m} \right|} \right), \quad (21)$$

$$v(a) = \frac{1}{2} \exp \left\{ \pm i \int_{a_I}^a m da \right\} \left(\sqrt{\left| \frac{m}{m_I} \right|} - \sqrt{\left| \frac{m_I}{m} \right|} \right), \quad (22)$$

where $m = m(a)$, and $m_I = m(a_I)$. In this manner, in proposed approach *quantum gravity is defined by monodromy between bases in the Fock space*

$$B_a = \begin{bmatrix} \left(\sqrt{\left| \frac{m_I}{m(a)} \right|} + \sqrt{\left| \frac{m(a)}{m_I} \right|} \right) \frac{e^{-i\lambda(a)}}{2} & \left(\sqrt{\left| \frac{m_I}{m(a)} \right|} - \sqrt{\left| \frac{m(a)}{m_I} \right|} \right) \frac{e^{i\lambda(a)}}{2} \\ \left(\sqrt{\left| \frac{m_I}{m(a)} \right|} - \sqrt{\left| \frac{m(a)}{m_I} \right|} \right) \frac{e^{-i\lambda(a)}}{2} & \left(\sqrt{\left| \frac{m_I}{m(a)} \right|} + \sqrt{\left| \frac{m(a)}{m_I} \right|} \right) \frac{e^{i\lambda(a)}}{2} \end{bmatrix} B_I. \quad (23)$$

where for compact notation $\lambda(a) = \pm \int_{a_I}^a m(a') da'$. The field operator $\hat{\Psi}$ that represents the Spacetime (3) as boson and defines quantum states of the manifold is

$$\hat{\Psi}[a] = \frac{1}{2m(a)} \sqrt{\frac{m_I}{2}} \left(e^{-i\lambda(a)} w_I + e^{i\lambda(a)} w_I^\dagger \right). \quad (24)$$

ONE-PARTICLE THERMODYNAMICS OF QUANTUM STATES

Initial data given by the basis (19) define thermal equilibrium state for ensemble of quantum states of the manifold (3). Let us consider thermal properties of this Spacetime - the final step of the Thermodynamical Einstein's Dream. Here we will study one-particle approximation [8] defined by density functional ρ_G in initial data basis

$$\rho_G = G^\dagger[a] G[a] = B_a^\dagger \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} B_a = B_I^\dagger \begin{bmatrix} |u|^2 & -uv \\ -u^* v^* & |v|^2 \end{bmatrix} B_I \equiv B_I^\dagger \rho_{eq} B_I. \quad (25)$$

Equilibrium entropy S and partition function Ω_{eq} of the system in initial data basis are

$$S = - \frac{\text{tr}(\rho_{eq} \ln \rho_{eq})}{\text{tr} \rho_{eq}} \equiv \ln \Omega_{eq}, \quad \Omega_{eq} = \frac{1}{2|u|^2 - 1}. \quad (26)$$

Physical quantum states of considered manifold are assumed to be described in the Gibbs ensemble. One can compute directly the entropy of the system

$$S = -\ln \langle n \rangle, \quad \langle n \rangle = 2n + 1. \quad (27)$$

Here $\langle n \rangle$ is averaged occupation number n of quantum states determined by

$$n \equiv \langle 0 | G^\dagger[a] G[a] | 0 \rangle = \frac{1}{4} \left(\left| \frac{m}{m_I} \right| + \left| \frac{m_I}{m} \right| \right) - \frac{1}{2}, \quad (28)$$

that together with natural conditions $n \geq 0$ and $|m| \geq |m_I|$ leads to the *energy spectrum*

$$\left| \frac{m}{m_I} \right| = \left| \frac{mc^2}{m_I c^2} \right| = \langle n \rangle + \sqrt{\langle n \rangle^2 - 1}, \quad (29)$$

and internal energy U and chemical potential μ computed by proper averaging are

$$U = \left[1 + \left(2 + \frac{1}{\langle n \rangle} \right) (\langle n \rangle - 1) \right] \left(\langle n \rangle + \sqrt{\langle n \rangle^2 - 1} \right) \frac{|m_I|}{2}, \quad (30)$$

$$\mu = \left[1 + \frac{1}{\langle n \rangle^2} - \left(2 - \frac{1}{\langle n \rangle} \right) \sqrt{\frac{1}{\langle n \rangle^2 - 1}} \right] \left(\langle n \rangle + \sqrt{\langle n \rangle^2 - 1} \right) |m_I|. \quad (31)$$

From bosonic property one can assume in equilibrium the Bose–Einstein statistics

$$\Omega_{\text{eq}} \equiv \left(\exp \left\{ \frac{U - \mu n}{T} \right\} - 1 \right)^{-1}, \quad (32)$$

and by this one can compute directly temperature of the system

$$T = \frac{\langle n \rangle + \sqrt{\langle n \rangle^2 - 1}}{\ln(\langle n \rangle + 1)} \left[1 + \left(1 - \frac{1}{\langle n \rangle} \right)^2 + \left(\langle n \rangle - \frac{1}{2\langle n \rangle} \right) \sqrt{\frac{\langle n \rangle - 1}{\langle n \rangle + 1}} \right] \frac{|m_I|}{2}. \quad (33)$$

Maximal entropy quantum states (MEQS) of the Riemannian manifold are defined by

$$S_{\text{max}} = 0 \implies n = 0, \quad \langle n \rangle = 1, \quad m = m_I, \quad U_{\text{max}} = \frac{|m_I|}{2}, \quad \mu_{\text{max}} = -\infty, \quad T_{\text{max}} = \frac{U_{\text{max}}}{\ln 2}. \quad (34)$$

These allow conclude that *for the metrics MEQS are fully determinated by initial data*. The last equation in (34) is an equation of state for MEQS, that describes classical ideal Bose gas. Infinite value of chemical potential means that MEQS create closed system, and the point $n = 0$ is phase transition point for MEQS set. Initial data point can be interpreted as a birth-point of the Riemannian manifold. Measurement of MEQS formally defines initial data point, and it defines initial data reference frame. Furthermore, MEQS set have very interesting physical interpretation that is a conclusion of the reasoning

$$m_I = m = i2V_0 a^2 H(a) \implies H(a) = \frac{Q}{a^2}, \quad Q = \frac{|m_I|}{2V_0} = \frac{U_{\text{max}}}{V_0} = a_I^2 H(a_I), \quad (35)$$

and for $U_{\text{max}} = p_{\text{max}} V_0$ it gives pressure $p_{\text{max}} = a_I^2 H(a_I)$. Form of the Hubble parameter $H(a)$ in (35) means that *set of MEQS for the manifold describes primordial radiation*. Initial data basis B_I is related with this light. In light of modern observational cosmology

one can identify this primordial light with CMB radiation. One can see that λ in (23) is linear for MEQS and fulfills the d'Alembert equation

$$\lambda(a) = \int_{a_I}^a m(a') da' \Big|_{m=m_I} = m_I(a - a_I) \Rightarrow \Delta\lambda = 0, \quad a(t) = a_I \sqrt{1 + 2Q(t - t_I)} \quad (36)$$

and by this can be treated as the Weyl scale [13] of the ensemble.

Quantum Gravity defined by the second quantization (13) used to the metrics (3) gives beautiful scenario for physics of the Universe. It allows to state that TED is realizable for considered manifold. From the String Theory point of view *tachyon is mass groundstate of bosonic string that describes some wider quantum theory of gravity*. Basing on this let one create the following conjecture: *for the Einstein–Hilbert (2) gravitational fields analogical deduction can be applied*. Moreover, multiparticle reasoning creates understanding of gravity by quantum field theory, and describes General Relativity by the Bose condensates. This approach can be called Multiparticle Quantum Gravity/Cosmology.

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